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## Contractions of Lie algebras and separation of variables

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**Abstract.** The Wigner–Inonu contraction from the rotation group  $O(3)$  to the Euclidean group  $E(2)$  is used to relate the separation of variables in the Laplace–Beltrami operators on two corresponding homogeneous spaces. Different realizations of the contraction take the two separable coordinate systems on the sphere  $S_2$  to the four on the plane  $E_2$ .

**Résumé.** La contraction de Wigner–Inonu du groupe de rotation  $O(3)$  au groupe euclidien  $E(2)$  est utilisée pour établir une relation entre la séparation des variables dans les opérateurs de Laplace–Beltrami sur les espaces homogènes correspondants. Différentes réalisations de la contraction transforment les deux systèmes de coordonnées sur la sphère  $S_2$  en quatre sur le plan  $E_2$ .

### 1. Introduction

It is well known that an intimate relationship exists between the theory of special functions and Lie group theory [1–3]. Virtually all properties of large classes of special functions can be obtained from the representation theory of Lie groups, making use of the fact that the special functions occur as basis functions of irreducible representations, as matrix elements of transformation matrices, as Clebsch–Gordon coefficients, or in some other guise. Recently, the class of functions treatable by group theoretical and algebraic methods has been extended to the so-called  $q$ -special functions that have been related to quantum groups [4–8].

One very fruitful application of Lie theory in this context is the algebraic approach to the separation of variables in partial differential equations [9–15]. In this approach separable coordinate systems (for Laplace–Beltrami, Hamilton–Jacobi and other invariant partial differential equations) are characterized by complete sets of commuting second-order operators. These lie in the enveloping algebra of the Lie algebra of the isometry group (or in some cases conformal group) of the corresponding homogeneous space. We mention that the operator approach to separation of variables has also been extended to quantum groups and thus to the separation of variables in differential-difference equations [16].

A question that has so far received little attention in the literature is that of the connections between the separation of variables in different spaces, e.g. in homogeneous spaces of different Lie groups. In particular, it is of interest to study the behaviour of separable coordinates, sets of commuting operators and the corresponding separating eigenfunctions under deformations and contractions of the underlying Lie algebras.

Two types of Lie algebra contractions exist in the literature. The first are standard Wigner–Inonu contractions [17–19]. They can be interpreted as singular limits of transformations of bases of Lie algebras. More recently, ‘graded contractions’ have been introduced [20–22]. They are more general than the Wigner–Inonu ones and can be obtained by introducing parameters that modify the structure constants of a Lie algebra  $L$  in a manner respecting a certain grading and then taking limits when these parameters go to zero.

Our aim is to perform a study of the connection between the contractions of Lie algebras and the separation of variables. In this first paper we restrict ourselves to the simplest case. We shall consider Wigner–Inonu contractions of the rotation algebra  $\mathfrak{o}(3)$  to the Euclidean algebra  $\mathfrak{e}(2)$ . The two separable coordinate systems on the sphere  $S_2 \sim \mathfrak{o}(3)/\mathfrak{o}(2)$  will be related to the four separable systems on the plane  $E_2 \sim \mathfrak{e}(2)/\mathfrak{o}(2)$ . The contractions will be followed through on several levels: the Lie algebra, the commuting sets of operators, the coordinate systems and the eigenfunctions of the Laplace–Beltrami operators.

Our motivation comes from several directions. Among them we mention the following. In special function theory contractions provide the possibility of obtaining new asymptotic formulas, new expansions, etc. In the theory of finite-dimensional integrable systems contractions provide relations between such systems in curved and flat spaces. Contractions play a significant role in the theory of quantum groups [23–25] and it is to be expected that methods developed for Lie groups will be generalizable to the case of quantum groups.

In section 2 we first review the two separable systems on the sphere  $S_2$  and the four on the plane  $E_2$ . We then introduce geodesical coordinates on  $S_2$  that are well adapted for the contraction limit. Using these coordinates we take the limit  $R \rightarrow \infty$ , where  $R$  is the radius of the sphere. Spherical coordinates on  $S_2$  go into polar or Cartesian ones on  $E_2$ . Elliptic coordinates on  $S_2$  go into elliptic, parabolic or Cartesian ones on  $E_2$ . Section 3 is devoted to the contraction of basis functions. Thus, spherical harmonics go over into Bessel functions or exponentials. Elliptic harmonics, expressed as products of Lamé polynomials, go into exponentials, Mathieu functions, or parabolic cylinder functions.

## 2. Complete sets of commuting operators, separable coordinates, and their contractions

### 2.1. Separable coordinates on the sphere $S_2$

Let us first consider the sphere  $S_2$ . Its isometry group is  $\mathfrak{o}(3)$ . We choose a standard basis  $\{L_1, L_2, L_3\}$  for the Lie algebra  $\mathfrak{o}(3)$ :

$$L_i = -\epsilon_{ikj} u_k \frac{\partial}{\partial u_j} \quad [L_i, L_k] = \epsilon_{ikj} L_j \quad i, k = 1, 2, 3 \quad (2.1)$$

where  $u_i$  are Cartesian coordinates in the ambient space  $E_3$ .

On the sphere  $S_2$  we have

$$u_1^2 + u_2^2 + u_3^2 = R^2.$$

The Laplace–Beltrami operator and metric on  $S_2$  in curvilinear coordinates are

$$\Delta_{\text{LB}} = \frac{1}{R^2} (L_1^2 + L_2^2 + L_3^2) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \sqrt{g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \quad (2.2)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad g = \det(g_{\mu\nu}) \quad g_{\alpha\mu} g^{\mu\nu} = \delta_\alpha^\nu.$$

Following the general method [9–15] (that has in particular been applied to the sphere

$S_2$ ) [26] we look for separated eigenfunctions of the Laplace–Beltrami operator satisfying

$$\Delta_{LB}\Psi = \frac{l(l+1)}{R^2}\Psi \quad X\Psi = k\Psi \quad \Psi_{lk}(\alpha, \beta) = \Xi_{lk}(\alpha)\Phi_{lk}(\beta) \quad (2.3)$$

where  $X$  is a second-order operator in the enveloping algebra of  $\mathfrak{o}(3)$ :

$$X = a_{ik}L_iL_k \quad a_{ik} = a_{ki}. \quad (2.4)$$

Two operators  $X$  and  $X'$  will be considered equivalent  $X \sim X'$ , if they are related by a rotation and a linear combination with the Laplacian

$$X \sim X' = (g^T a g)_{ik} L_i L_k + \mu \Delta \quad g^T g = I. \quad (2.5)$$

The matrix  $a_{ik}$  can be diagonalized to give

$$X(a_1, a_2, a_3) \equiv X = a_1 L_1^2 + a_2 L_2^2 + a_3 L_3^2. \quad (2.6)$$

For  $a_1 = a_2 = a_3$  we have  $X \sim 0$ . If two eigenvalues of  $a_{ik}$  are equal, e.g.  $a_1 = a_2 \neq a_3$ , we can transform  $X$  into  $X(0, 0, 1) = L_3^2$  and the corresponding separable coordinates on  $S_2$  are the usual spherical ones

$$u_1 = R \sin \vartheta \cos \varphi \quad u_2 = R \sin \vartheta \sin \varphi \quad u_3 = R \cos \vartheta. \quad (2.7)$$

They correspond to the group reduction  $\mathfrak{o}(3) \supset \mathfrak{o}(2)$  and  $X = L_3^2$  is invariant under  $\mathfrak{o}(2)$  and under reflections in all coordinate planes.

When all three eigenvalues  $a_i$  are different, then the separable coordinates in (2.3) are elliptic ones [26–29]. These can be written in algebraic form, as

$$\begin{aligned} u_1^2 &= R^2 \frac{(\rho_1 - a_1)(\rho_2 - a_1)}{(a_2 - a_1)(a_3 - a_1)} \\ u_2^2 &= R^2 \frac{(\rho_1 - a_2)(\rho_2 - a_2)}{(a_3 - a_2)(a_1 - a_2)} \\ u_3^2 &= R^2 \frac{(\rho_1 - a_3)(\rho_2 - a_3)}{(a_1 - a_3)(a_2 - a_3)} \end{aligned} \quad (2.8)$$

with  $a_1 \leq \rho_1 \leq a_2 \leq \rho_2 \leq a_3$ .

In trigonometric form we put

$$\rho_1 = a_1 + (a_2 - a_1) \cos^2 \phi \quad \rho_2 = a_3 - (a_3 - a_2) \cos^2 \theta \quad (2.9)$$

and obtain

$$\left. \begin{aligned} u_1 &= R \sqrt{1 - k'^2 \cos^2 \theta} \cos \phi \\ u_2 &= R \sin \theta \sin \phi \\ u_3 &= R \sqrt{1 - k^2 \cos^2 \phi} \cos \theta \end{aligned} \right\} \quad 0 \leq \phi < 2\pi \quad 0 \leq \theta \leq \pi \quad (2.10)$$

where

$$k^2 = \frac{a_2 - a_1}{a_3 - a_1} = \sin^2 f \quad (k')^2 = \frac{a_3 - a_2}{a_3 - a_1} = \cos^2 f. \quad (2.11)$$

Finally, the Jacobi elliptic version of elliptic coordinates is obtained by putting

$$\rho_1 = a_1 + (a_2 - a_1) \operatorname{sn}^2(\alpha, k) \quad \rho_2 = a_2 + (a_3 - a_2) \operatorname{cn}^2(\beta, k'). \quad (2.12)$$

We obtain

$$\left. \begin{aligned} u_1 &= R \operatorname{sn}(\alpha, k) \operatorname{dn}(\beta, k') \\ u_2 &= R \operatorname{cn}(\alpha, k) \operatorname{cn}(\beta, k') \\ u_3 &= R \operatorname{dn}(\alpha, k) \operatorname{sn}(\beta, k'). \end{aligned} \right\} \quad -K \leq \alpha \leq K \quad -2K' \leq \beta \leq 2K' \quad (2.13)$$

where  $\operatorname{sn}(\alpha, k)$ ,  $\operatorname{cn}(\alpha, k)$  and  $\operatorname{dn}(\beta, k)$  are the Jacobi elliptic functions with modulus  $k$  and  $K$  and  $K'$  are the complete elliptic integrals [30, 31]. The moduli are given in (2.11), we have  $k^2 + k'^2 = 1$  and  $2fR$  is the interfocal distance for the ellipses on the upper hemisphere.

Elliptic coordinates corresponding to the reduction  $O(3) \supset D_2$ , where  $D_2$  is the dihedral group (rotations through  $\pi$  about all three axes and reflections in a coordinate plane). Indeed, the operator (2.6) is invariant only under  $D_2$ , rather than  $O(2)$ , for  $a_1 \neq a_2 \neq a_3 \neq a_1$ .

## 2.2. Separable coordinates on the Euclidean plane $E_2$

Let us consider the Lie algebra  $e(2)$  in the basis

$$L_3 = u_2 \partial_{u_1} - u_1 \partial_{u_2} \quad P_1 = \partial_{u_1} \quad P_2 = \partial_{u_2}. \quad (2.14)$$

Separated eigenfunctions of the Laplace operator  $\Delta = P_1^2 + P_2^2$  satisfy

$$\Delta \Phi_{k,\lambda} = k^2 \Phi_{k,\lambda} \quad X \Phi_{k,\lambda} = \lambda \Phi_{k,\lambda} \quad \Phi_{k,\lambda}(\alpha, \beta) = \Sigma_{k,\lambda}(\alpha) \Psi_{k,\lambda}(\beta) \quad (2.15)$$

where  $X$  is the second-order operator

$$X = aL_3^2 + b(L_3P_1 + P_1L_3) + c(L_3P_2 + P_2L_3) + dP_1^2 + eP_2^2 + 2fP_1P_2. \quad (2.16)$$

By means of Euclidean transformations, and linear combination with  $\Delta$ , we can take  $X$  into precisely one of the following operators

$$X_S = L_3^2 \quad (a \neq 0, D = 0) \quad (2.17)$$

$$X_C = P_1^2 - P_2^2 \quad (a = b = c = 0) \quad (2.18)$$

$$X_P = L_3P_1 + P_1L_3 \quad (a = 0, b^2 + c^2 \neq 0) \quad (2.19)$$

$$X_E = L_3^2 + \frac{1}{2}D^2(P_1^2 - P_2^2) \quad (a \neq 0, D \neq 0) \quad (2.20)$$

where

$$D^2 = \frac{1}{2a^2} \{4b^2c^2 + [a(b-e) - b^2 + c^2]^{1/2}\}. \quad (2.21)$$

Each of the operators (2.17)–(2.20) corresponds to a different separable coordinate system in (2.15). Thus  $X_C$  corresponds to Cartesian coordinates  $(x, y)$ ,  $X_S$  to polar ones

$$x = \rho \cos \phi \quad y = \rho \sin \phi \quad (2.22)$$

$X_P$  to parabolic coordinates

$$x = \frac{1}{2}(u^2 - v^2) \quad y = uv \quad (2.23)$$

and  $X_E$  to elliptic ones

$$x = D \cosh \xi \cos \eta \quad y = D \sinh \xi \sin \eta \quad (2.24)$$

where  $2D$  is the focal distance.

2.3. The contractions

We shall use  $R^{-1}$  as the contraction parameter. To realize the contraction explicitly, let us introduce geodesical coordinates on the sphere [32], putting

$$x_\mu = R \frac{u_\mu}{u_3} = \frac{u_\mu}{\sqrt{1 - (u_1^2 + u_2^2)/R^2}} \quad \mu = 1, 2. \tag{2.25}$$

The  $\mathfrak{o}(3)$  generators can then be expressed as

$$-\frac{L_1}{R} \equiv \pi_2 = p_2 + \frac{1}{R^2} x_2(x_1 p_1 + x_2 p_2) \tag{2.26}$$

$$\frac{L_2}{R} \equiv \pi_1 = p_1 + \frac{1}{R^2} x_1(x_1 p_1 + x_2 p_2) \tag{2.27}$$

$$L_3 = x_2 p_1 - x_1 p_2 = x_2 \pi_1 - x_1 \pi_2. \tag{2.28}$$

The commutation relations are

$$[L_3, \pi_1] = \pi_2 \quad [L_3, \pi_2] = -\pi_1 \quad [\pi_1, \pi_2] = \frac{L_3}{R^2} \tag{2.29}$$

so that for  $R \rightarrow \infty$  the  $\mathfrak{o}(3)$  algebra contracts to the  $\mathfrak{e}(2)$  algebra. Moreover the momenta  $\pi_\mu$  contract to  $p_\mu = \partial/\partial x_\mu$  ( $\mu = 1, 2$ ).

The  $\mathfrak{o}(3)$  Laplace–Beltrami operator (2.2) contracts to the  $\mathfrak{e}(2)$  operator:

$$\Delta_{LB} = \pi_1^2 + \pi_2^2 + \frac{L_3^2}{R^2} \rightarrow \Delta = (p_1^2 + p_2^2). \tag{2.30}$$

Let us now consider the contractions of the operator (2.6) and of the corresponding coordinates.

2.3.1. Spherical coordinates on  $S_2$  to polar coordinates on  $E_2$ . We choose  $a_1 = a_2$  in (2.6) and put

$$\tan \theta = \frac{r}{R}. \tag{2.31}$$

In the limit  $R \rightarrow \infty$ ,  $\theta \rightarrow 0$  we have

$$X = L_3^2 \rightarrow X_S = L_3^2 \tag{2.32}$$

and

$$\begin{aligned} x_1 &= R \frac{u_1}{u_3} \rightarrow x = r \cos \phi \\ x_2 &= R \frac{u_2}{u_3} \rightarrow y = r \sin \phi. \end{aligned} \tag{2.33}$$

2.3.2. Spherical coordinates on  $S_2$  to Cartesian on  $E_2$ . We choose  $a_2 = a_3 \sim 0$  in (2.6) so that the coordinates (2.7) permute into

$$u_1 = R \cos \theta' \quad u_2 = R \sin \theta' \cos \phi' \quad u_3 = R \sin \theta' \sin \phi'. \tag{2.34}$$

Putting

$$\cos \theta' = \frac{x}{R} \quad \cos \phi' = \frac{y}{R} \tag{2.35}$$

and taking the limit  $R \rightarrow \infty$ ,  $\theta' \rightarrow \frac{\pi}{2}$ ,  $\phi' \rightarrow \frac{\pi}{2}$ , we obtain

$$X\left(\frac{1}{R^2}, 0, 0\right) = \frac{L_1^2}{R^2} = \pi_1^2 \rightarrow X = P_1^2 \sim X_C \tag{2.36}$$

and

$$x_1 = R \frac{\cot \theta'}{\sin \phi'} \rightarrow x \quad x_2 = R \cot \phi' \rightarrow y. \tag{2.37}$$

*2.3.3. Elliptic coordinates on  $S_2$  to elliptic coordinates on  $E_2$ .* We take  $X$  in its general form, equivalent to

$$X = L_3^2 - \left(\frac{a_2 - a_1}{a_3 - a_2}\right) L_1^2. \tag{2.38}$$

We put

$$\frac{R^2}{a_3 - a_1} = \frac{D^2}{a_2 - a_1} \quad L_1 = -R\pi_2 \tag{2.39}$$

and in the limit  $R^2 \sim a_3 \rightarrow \infty$  obtain

$$X \rightarrow L_3^2 - D^2 P_2^2 \sim X_E. \tag{2.40}$$

For the coordinates we put

$$\rho_1 = a_1 + (a_2 - a_1) \cos^2 \eta \quad \rho_2 = a_1 + (a_2 - a_1) \cosh^2 \xi \tag{2.41}$$

and for  $R^2 \sim a_3 \rightarrow \infty$ , using (2.39), we obtain (2.24), i.e. elliptic coordinates on the plane  $E_2$ .

*2.3.4. Elliptic coordinates on  $S_2$  to Cartesian coordinates on  $E_2$ .* We start from the coordinates (2.8) but change the ordering of the parameters  $a_i$  i.e. put

$$a_1 \leq \rho_1 \leq a_3 \leq \rho_2 \leq a_2 \tag{2.42}$$

and choose  $a_3 - a_1 = a_2 - a_3 \equiv a$ . We then have

$$\frac{1}{aR^2} X = (\pi_1^2 - \pi_2^2) \rightarrow P_1^2 - P_2^2. \tag{2.43}$$

For the coordinates we put

$$\frac{a_3 - \rho_1}{a} = \xi_1 \quad \frac{\rho_2 - a_3}{a} = \xi_2. \tag{2.44}$$

Using equations (2.25) and (2.8) we have

$$x_1^2 = R^2 \frac{(1 - \xi_1)(1 + \xi_2)}{2\xi_1\xi_2} \quad x_2^2 = R^2 \frac{(1 + \xi_1)(1 - \xi_2)}{2\xi_1\xi_2}. \tag{2.45}$$

From equation (2.45) we obtain

$$\xi_{1,2} = \frac{R^2}{R^2 + x_1^2 + x_2^2} \left\{ \left[ 1 + \frac{x_1^2 + x_2^2}{R^2} + \frac{(x_1^2 - x_2^2)^2}{4R^4} \right]^{1/2} \mp \frac{x_1^2 - x_2^2}{2R^2} \right\}. \tag{2.46}$$

In the limit  $R \rightarrow \infty$  we have

$$\xi_1 \rightarrow 1 - \frac{x^2}{R^2} \quad \xi_2 \rightarrow 1 - \frac{y^2}{R^2} \tag{2.47}$$

and hence  $x_1$  and  $x_2$  of (2.45) go into Cartesian coordinates:

$$x_1 \rightarrow x \quad x_2 \rightarrow y. \tag{2.48}$$

2.3.5. *Elliptic coordinates on  $S_2$  to parabolic coordinates on  $E_2$ .* We take the operator (2.6) with  $a_1 \leq \rho_1 \leq a_2 \leq \rho_2 \leq a_3$  and choose  $a_3 - a_2 = a_2 - a_1 \equiv a$ . We must first ‘undo’ the diagonalization (2.5) by a rotation through  $\pi/4$ . The operator (2.6) transforms into

$$\frac{1}{aR} X_S = -\frac{1}{aR} (L_1 L_3 + L_3 L_1) = L_3 \pi_2 + \pi_2 L_3 \tag{2.49}$$

with the correct limit for  $R \rightarrow \infty$ . The coordinates (2.13) on  $S_2$  are rotated into

$$\begin{aligned} u_1 &= \frac{R}{\sqrt{2}} (\operatorname{sn} \alpha \operatorname{dn} \beta + \operatorname{dn} \alpha \operatorname{sn} \beta) \\ u_2 &= R \operatorname{cn} \alpha \operatorname{cn} \beta \\ u_3 &= \frac{R}{\sqrt{2}} (\operatorname{dn} \alpha \operatorname{sn} \beta - \operatorname{sn} \alpha \operatorname{dn} \beta) \end{aligned} \tag{2.50}$$

with modulus  $k = k' = 1/\sqrt{2}$  for all Jacobi elliptic functions.

From equation (2.50) we obtain

$$\begin{aligned} \operatorname{sn} \alpha &= \frac{1}{\sqrt{2}} \left[ \left(1 + \frac{u_1}{R}\right)^{1/2} \left(1 - \frac{u_3}{R}\right)^{1/2} - \left(1 - \frac{u_1}{R}\right)^{1/2} \left(1 + \frac{u_3}{R}\right)^{1/2} \right] \\ \sqrt{2} \operatorname{dn} \beta &= \frac{1}{\sqrt{2}} \left[ \left(1 + \frac{u_1}{R}\right)^{1/2} \left(1 - \frac{u_3}{R}\right)^{1/2} + \left(1 - \frac{u_1}{R}\right)^{1/2} \left(1 + \frac{u_3}{R}\right)^{1/2} \right]. \end{aligned} \tag{2.51}$$

Equation (2.51) suggest the limiting procedure. Indeed we put

$$\operatorname{sn} \alpha = -1 + \frac{u^2}{2R} \quad \sqrt{2} \operatorname{dn} \beta = 1 + \frac{v^2}{2R}. \tag{2.52}$$

In the limit  $R \rightarrow \infty$  we than obtain

$$x_1 \rightarrow x = \frac{u^2 - v^2}{2} \quad x_2 \rightarrow y = uv \tag{2.53}$$

i.e. the parabolic coordinates (2.23).

### 3. Contraction of basis functions

Having established the contraction properties of separable coordinates and the corresponding complete sets of commuting operators. We shall now consider the behaviour of eigenfunctions.

#### 3.1. Spherical basis on $S_2$ to polar basis on $E_2$

We start from the standard spherical functions  $Y_{lm}(\theta, \phi)$  as basis functions of irreducible representations of the group  $O(3)$  (see, e.g., [33])

$$\begin{aligned} \Psi_{lm}(\theta, \phi) &= \frac{1}{\sqrt{R}} Y_{lm}(\theta, \phi) = \frac{(-1)^{(m+|m|)/2}}{\sqrt{R}} \left[ \frac{2l+1}{2} \frac{(l+|m|)!}{(l-|m|)!} \right]^{1/2} \frac{(\sin \theta)^{|m|}}{2^{|m|} |m|!} \\ &\quad \times {}_2F_1 \left( -l+|m|, l+|m|+1; |m|+1; \sin^2(\frac{1}{2}\theta) \right) \frac{e^{im\phi}}{\sqrt{2\pi}}. \end{aligned} \tag{3.1}$$

In the contraction limit  $R \rightarrow \infty$  we put

$$\tan \theta \sim \theta \sim \frac{r}{R} \quad l \sim kR. \tag{3.2}$$



Using the asymptotic formulae

$$\lim_{R \rightarrow \infty} {}_2F_1 \left( -kR, kR; |m| + 1; \frac{r^2}{4R^2} \right) = {}_0F_1 \left( |m| + 1; -\frac{k^2 r^2}{4} \right) \quad (3.3)$$

$$\lim_{z \rightarrow \infty} \frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha - \beta}$$

and the formula

$$J_\nu(z) = \left( \frac{z}{2} \right)^\nu \frac{1}{\Gamma(\nu + 1)} {}_0F_1 \left( \nu + 1; -\frac{z^2}{4} \right) \quad (3.4)$$

we obtain

$$\lim_{\substack{R \rightarrow \infty \\ \theta \rightarrow 0}} \frac{1}{\sqrt{R}} Y_{lm}(\theta, \phi) = (-1)^{(m+|m|)/2} \sqrt{k} J_{|m|}(kr) \frac{e^{im\phi}}{\sqrt{2\pi}}. \quad (3.5)$$

The result (3.5) is not new [33]. The point is that this asymptotic formula is obtained very naturally in the context of group contractions applied to the separation of variables.

### 3.2. Spherical basis on $S_2$ to Cartesian basis on $E_2$

We start from the coordinates (2.34), but drop the primes, and write the corresponding spherical functions as

$$Y_{lm}(\theta, \phi) = \frac{\sqrt{2l+1}}{2\pi} e^{im\phi} (\sin \theta)^{|m|} \times \begin{cases} (-1)^{(l+m)/2} \left[ \frac{\Gamma(\frac{l+m+1}{2})\Gamma(\frac{l-m+1}{2})}{\Gamma(\frac{l+m+2}{2})\Gamma(\frac{l-m+2}{2})} \right]^{\frac{1}{2}} F \left( -\frac{l-m}{2}, \frac{l+m+1}{2}; \frac{1}{2}; \cos^2 \theta \right) \\ (-1)^{(l+m-1)/2} \left[ \frac{\Gamma(\frac{l+m+2}{2})\Gamma(\frac{l-m+2}{2})}{\Gamma(\frac{l+m+1}{2})\Gamma(\frac{l-m+1}{2})} \right]^{\frac{1}{2}} 2 \cos \theta F \left( -\frac{l-m-1}{2}, \frac{l+m+2}{2}; \frac{3}{2}; \cos^2 \theta \right) \end{cases} \quad (3.6)$$

for  $l+m$  even and odd, respectively, we now put

$$l \sim kR \quad m \sim k_2R \quad (3.7)$$

and

$$\sin \theta \rightarrow 1 \quad \cos \theta \rightarrow \frac{x}{R} \quad \phi \rightarrow \frac{y}{R} - \frac{\pi}{2}. \quad (3.8)$$

The  $F \equiv {}_2F_1$  hypergeometric functions simplify to  ${}_0F_1$  ones, the  $\Gamma$  functions also simplify and we find

$$Y_{lm}(\theta, \phi) \xrightarrow{R \rightarrow \infty} (-1)^{m/2} \sqrt{2kR} \frac{e^{ik_2y}}{2\pi} \begin{cases} (-1)^{(l+m)/2} \left( \frac{2}{k_1R} \right)^{\frac{1}{2}} {}_0F_1 \left( \frac{1}{2}; \frac{-k_1^2 x^2}{4} \right) \\ (-1)^{(l+m-1)/2} \frac{2x}{R} \left( \frac{k_1R}{2} \right)^{\frac{1}{2}} {}_0F_1 \left( \frac{3}{2}; \frac{-k_1^2 x^2}{4} \right) \end{cases} \quad (3.9)$$

with  $k_1^2 + k_2^2 = k^2$ .

The  ${}_0F_1$  functions are in this case expressible in terms of the Bessel functions  $J_{\pm 1/2}$  that are, essentially, trigonometric functions. The final result is that under the contraction we have

$$\lim_{R \rightarrow \infty} (-1)^{l/2} Y_{lm}(\theta, \phi) = \frac{e^{ik_2y}}{\sqrt{2\pi}} \begin{cases} \cos k_1x \\ -i \sin k_1x \end{cases} \quad (3.10)$$

for  $l + m$  even and odd, respectively, where the limit is taken as in (3.7). The parity properties of  $Y_{lm}$  under the exchange  $\theta \rightarrow \pi - \theta$  have led to the appearance of  $\cos k_1x$  and  $\sin k_1x$  in (3.9), instead of the usual Cartesian coordinate solution  $\exp i(k_1x + k_2y)$ .

### 3.3. Solutions of the Lamé equation

Let us consider (2.3) on the sphere  $S_2$  and separate variables in the elliptic coordinates (2.8). We obtain two ordinary differential equation of the form

$$\frac{d^2\psi}{d\rho^2} + \frac{1}{2} \left\{ \frac{1}{\rho - a_1} + \frac{1}{\rho - a_2} + \frac{1}{\rho - a_3} \right\} \frac{d\psi}{d\rho} + \frac{1}{4} \left\{ \frac{\lambda - l(l + 1)\rho}{(\rho - a_1)(\rho - a_2)(\rho - a_3)} \right\} \psi = 0 \tag{3.11}$$

or equivalently

$$4\sqrt{P(\rho)} \frac{d}{d\rho} \sqrt{P(\rho)} \frac{d\psi}{d\rho} - \{l(l + 1)\rho - \lambda\} \psi = 0 \tag{3.12}$$

where

$$P(\rho) = (\rho - a_1)(\rho - a_2)(\rho - a_3).$$

Equation (3.11) is the Lamé equation in algebraic form. It is a Fuchsian type equation with four regular singularities (at  $a_1, a_2, a_3$  and  $\infty$ ) [26–30, 34].

Its general solution can be represented by a series expansion about any one of the singular points  $a_k$  as

$$\psi(\rho) = (\rho - a_1)^{\alpha_1/2} (\rho - a_2)^{\alpha_2/2} (\rho - a_3)^{\alpha_3/2} \sum_{t=0}^{\infty} b_t^{(k)} (\rho - a_k)^t \tag{3.13}$$

where we have

$$\alpha_j(\alpha_j - 1) = 0 \quad j = 1, 2, 3$$

and can choose  $k$  equal to 1, 2, or 3.

Substituting in the Lamé equation (3.11) we obtain a three-term recursion relation for  $b_t^k$ :

$$\beta_t^k b_{t+1}^k + [\gamma_t^k + \lambda - l(l + 1)a_k] b_t^k + (2t + \alpha - l - 2)(2t + \alpha + l - 1) b_{t-1}^k = 0 \tag{3.14}$$

with

$$\begin{aligned} \alpha &= \alpha_1 + \alpha_2 + \alpha_3 & \alpha_{ik} &= \alpha_i - \alpha_k & b_{-1} &= 0 \\ \beta_t^{(k)} &= 4(a_i - a_k)(a_j - a_k)(t + 1)(t + \alpha_k + \frac{1}{2}) & & & & (i, j, k \text{ cyclic}) \\ \gamma_t^{(k)} &= -(a_i - a_k)(2t + \alpha_k + \alpha_j)^2 - (a_j - a_k)(2t + \alpha_k + \alpha_i)^2. \end{aligned} \tag{3.15}$$

The expansion (3.13) represents a Lamé function. Since we are interested in representations of  $SO(3)$ , the sum in  $\psi(\rho)$  must be a polynomial of order  $N$ , i.e. we must have

$$b_N \neq 0 \quad b_{N+1} = b_{N+2} = \dots = 0$$

for some  $N$ . The condition for this is that we have

$$l = 2N + \alpha \tag{3.16}$$

and we obtain a secular equation for the eigenvalues  $\lambda$ , i.e. the separation constant in elliptic coordinates, by requiring that the determinant of the homogeneous linear system (3.14) for  $\{b_0, b_1, \dots, b_N\}$  should vanish. Since  $N$  and  $l$  must be integers, equation (3.16) implies that  $\alpha$  and  $l$  must have the same parity.

Numerous further properties of the Lamé polynomials, in the context of representations of the group  $O(3)$ , in the  $O(3) \supset D_2$  basis, were established, e.g., in [26-29].

Here let us just represent the basis functions as

$$\Psi_{l\lambda}^{pq}(\rho_1, \rho_2) = A_{l\lambda}^{pq} \psi_{l\lambda}^{pq}(\rho_1) \psi_{l\lambda}^{pq}(\rho_2) \tag{3.17}$$

where  $A_{l\lambda}^{pq}$  is some normalization constant. The labels  $p, q$  take values  $\pm 1$  and identify representations of  $D_2$ . For each value of  $l$  the values of  $p, q$  and  $\lambda$  label  $(2l + 1)$  different states. Since a given representations  $(p, q)$  of  $D_2$  can figure more than once in the reduction of a representation of  $O(3)$  corresponding to a given  $l$ , we are faced with a ‘missing label problem’, resolved by the quantum number  $\lambda$ , i.e. the operator  $X$  of (2.6).

The expansions that we shall use for the Lamé polynomials in (3.17) are as in (3.13), but the summation over  $t$  is from  $t = 0$  to  $t = N$ .

### 3.4. Elliptic basis on $S_2$ to Cartesian basis on $E_2$

We choose elliptic coordinates on  $S_2$  as in (2.8), but with  $a_1 < a_3 < a_2$ , as in (2.42). We write the basis functions as in (3.17) with

$$\begin{aligned} \psi_{l\lambda}(\rho_1) &= (\rho_1 - a_1)^{\alpha_1/2} (\rho_1 - a_2)^{\alpha_2/2} (\rho_1 - a_3)^{\alpha_3/2} \sum_{t=0}^N b_t^{(1)} (\rho_1 - a_1)^t \\ \psi_{l\lambda}(\rho_2) &= (\rho_2 - a_1)^{\alpha_1/2} (\rho_2 - a_2)^{\alpha_2/2} (\rho_2 - a_3)^{\alpha_3/2} \sum_{t=0}^N b_t^{(2)} (\rho_2 - a_2)^t \end{aligned} \tag{3.18}$$

as in (3.13). The coefficients  $b_t^j$  ( $j = 1, 2$ ) satisfy the recursion relation (3.14) and we have  $N = (l - \alpha)/2$ . We use the coordinates  $\xi_1$  and  $\xi_2$  introduced in (2.44) (for  $a \equiv a_3 - a_1 = a_2 - a_3$ ). Equation (3.18) reduces to

$$\begin{aligned} \psi_{l\lambda}(\xi_1) &= (-1)^{(\alpha_2 + \alpha_3)/2} a^{\alpha/2} (1 - \xi_1)^{\alpha_1/2} (1 + \xi_1)^{\alpha_2/2} \xi_1^{\alpha_3/2} \sum_{t=0}^N C_t^1 (1 - \xi_1)^t \\ \psi_{l\lambda}(\xi_2) &= (-1)^{\alpha_2/2} a^{\alpha/2} (1 - \xi_2)^{\alpha_2/2} (1 + \xi_2)^{\alpha_1/2} \xi_2^{\alpha_3/2} \sum_{t=0}^N C_t^2 (1 - \xi_2)^t \end{aligned} \tag{3.19}$$

with  $C_t^{(1)} = a^t b_t$ ,  $C_t^{(2)} = (-a)^t b_t$ .

The recursion relations (3.14) now imply

$$\begin{aligned} 8(t + 1)(t + \alpha_1 + \frac{1}{2})C_{t+1}^{(1)} + \{\mu^{(1)} - 2(2t + \alpha_1 + \alpha_3)^2 - (2t + \alpha_1 + \alpha_2)^2\}C_t^{(1)} \\ + (2t + \alpha - l - 2)(2t + \alpha + l - 1)C_{t-1}^{(1)} = 0 \\ -8(t + 1)(t + \alpha_2 + \frac{1}{2})C_{t+1}^{(2)} + \{\mu^{(2)} + 2(2t + \alpha_2 + \alpha_3)^2 + (2t + \alpha_1 + \alpha_2)^2\}C_t^{(2)} \\ - (2t + \alpha - l - 2)(2t + \alpha + l - 1)C_{t-1}^{(2)} = 0 \end{aligned} \tag{3.20}$$

where

$$\mu^{(j)} = \frac{1}{a} [\lambda - a_j l(l + 1)] \quad j = 1, 2. \tag{3.21}$$

The contraction limit is taken using equation (2.47) to relate  $\xi_{1,2}$  to the Cartesian coordinates on  $E_2$ . Taking  $l \sim kR$  we find

$$\mu^{(1)} \rightarrow 2R^2 k_1^2 \quad \mu^{(2)} \rightarrow -2R^2 k_2^2 \quad k = \sqrt{k_1^2 + k_2^2}. \tag{3.22}$$

For  $R \rightarrow \infty$  the recursion relations (3.20) simplify to two term ones that can be solved to obtain

$$C_t^{(j)} = \frac{R^{2t}}{(\alpha_j + \frac{1}{2})_t} \left( \frac{-k_j^2}{4} \right)^t \frac{1}{t!} \tag{3.23}$$

with

$$(\alpha_j + \frac{1}{2})_t = (\alpha_j + \frac{1}{2})(\alpha_j + \frac{3}{2}) \cdots (\alpha_j - \frac{3}{2} + t) \quad t \geq 1 \quad (\alpha_j + \frac{1}{2})_0 = 1.$$

Substituting (3.23) into (3.19) we obtain

$$\begin{aligned} \psi_{l\lambda}(\xi_1) &= (-1)^{(\alpha_2+\alpha_3)/2} \frac{a^{\alpha/2}}{R^{\alpha_1}} x^{\alpha_1} {}_0F_1 \left( \alpha_1 + \frac{1}{2}; -\frac{k_1^2 x^2}{4} \right) \\ \psi_{l\lambda}(\xi_2) &= (-1)^{\alpha_2/2} \frac{a^{\alpha/2}}{R^{\alpha_2}} y^{\alpha_2} {}_0F_1 \left( \alpha_2 + \frac{1}{2}; -\frac{k_2^2 y^2}{4} \right). \end{aligned} \tag{3.24}$$

Using equation (3.4) and the explicit expressions for  $J_{\pm 1/2}$ , we find the contraction limit:

$$A_{l\lambda}^{pq}(R) \psi_{l\lambda}(\xi_1, \xi_2) \rightarrow A_{l\lambda}^{pq} \psi_{k_1}(x) \psi_{k_2}(y)$$

$$= A_{l\lambda}^{pq} (-1)^{\alpha_3/2} a^{\alpha/2} \begin{cases} \cos k_1 x \cos k_2 y & \alpha_1 = 0, \alpha_2 = 0 \\ -\frac{1}{k_2 R} \cos k_1 x \sin k_2 y & \alpha_1 = 0, \alpha_2 = 1 \\ -\frac{1}{k_1 R} \sin k_1 x \cos k_2 y & \alpha_1 = 1, \alpha_2 = 0 \\ -\frac{1}{k_1 k_2 R^2} \sin k_1 x \sin k_2 y & \alpha_1 = 1, \alpha_2 = 1. \end{cases} \tag{3.25}$$

Though these formulae are quite simple, to our knowledge they are new.

### 3.5. Elliptic basis on $S_2$ to elliptic basis on $E_2$

Let us start from the elliptic coordinates (2.8) with  $a_1 \leq \rho_1 \leq a_2 \leq \rho_2 \leq a_3$ . We take the limit  $R \rightarrow \infty$ ,  $a_3 \rightarrow \infty$  with  $\sqrt{a_3}/R$ ,  $a_1$  and  $a_2$  finite. We introduce a constant  $D$  as in (2.39). Elliptic coordinates on the plane  $E_2$  are introduced via (2.41), so that the Cartesian coordinates  $(x, y)$  are expressed in terms of the elliptic ones  $(\xi, \eta)$  as in (2.24). Let us first take the limit in the separated equations (3.12). Going over to the variables  $(\xi, \eta)$  from  $(\rho_1, \rho_2)$ , for  $R \rightarrow \infty$  we obtain:

$$\frac{d^2 \psi_1}{d\eta^2} + \left\{ \mu - \frac{k^2 D^2}{2} \left( \frac{a_2 + a_1}{a_2 - a_1} \right) - \frac{k^2 D^2}{2} \cos 2\eta \right\} \psi_1 = 0 \tag{3.26}$$

$$\frac{d^2 \psi_2}{d\xi^2} + \left\{ \mu - \frac{k^2 D^2}{2} \left( \frac{a_2 + a_1}{a_2 - a_1} \right) - \frac{k^2 D^2}{2} \cosh 2\xi \right\} \psi_2 = 0 \tag{3.27}$$

with

$$\mu = \frac{\lambda}{a_3} \quad l \sim kR.$$

In equation (3.26) we recognize the standard form of the Mathieu equation, whereas equation (3.27) is a modified Mathieu equation [35]. Thus, in the contraction limit, Lamé functions will go over into Mathieu ones. Moreover, periodic solutions of the Lamé equation go over into periodic solutions of (3.26).

The contraction limit can also be taken directly in the Lamé polynomials, using the expansion (3.13) (cut off at  $t = N$ ). The result that we obtain is

$$\lim_{R \rightarrow \infty} \frac{\Psi_{l\lambda}(\rho_1)}{R^{\alpha_3}} = (a_2 - a_1)^{\alpha/2} \frac{(-1)^{(\alpha_2 + \alpha_3)/2}}{D^{\alpha_3}} (\cos \eta)^{\alpha_1} (\sin \eta)^{\alpha_2} \sum_{t=0}^{\infty} C_t (\cos \eta)^{2t} \quad (3.28)$$

$$\lim_{R \rightarrow \infty} \frac{\Psi_{l\lambda}(\rho_2)}{R^{\alpha_3}} = (a_2 - a_1)^{\alpha/2} \frac{(-1)^{\alpha_3/2}}{D^{\alpha_3}} (\cosh \xi)^{\alpha_1} (\sinh \xi)^{\alpha_2} \sum_{t=0}^{\infty} C_t (\cosh \xi)^{2t} \quad (3.29)$$

where the expansion coefficients  $C_t$  satisfy recursion relations obtained from (3.14), namely  $4(t+1)(t+1/2+\alpha_1)C_t + \{\mu - (2t+\alpha_1+\alpha_2)^2\}C_t - k^2 D^2 C_t = 0$ . (3.30)

Depending on the values of  $\alpha_1$  and  $\alpha_2$ , defined in (3.18), (3.30) takes one of four different forms.

### 3.6. Elliptic basis on $S_2$ to parabolic basis on $E_2$

Let us consider the contraction limit for the Lamé equations (3.12). To do this we use equations (2.12) with  $a_3 - a_2 = a_2 - a_1 = a$  i.e.  $k = k' = 1/\sqrt{2}$ , together with (2.52), to obtain

$$\rho_1 \sim a_1 + a \left( -1 + \frac{u^2}{2R} \right) \quad \rho_2 \sim a_1 + a \left( 1 + \frac{v^2}{R} \right). \quad (3.31)$$

Equation (3.12) for  $\rho = \rho_1$  and  $\rho = \rho_2$  in the limit  $R \rightarrow \infty$ , with  $l^2 \sim k^2 R^2$  and  $\lambda - a_2 l(l+1) = \mu R a$ , yields the two equations

$$\frac{d^2 \psi_1}{du^2} + (k^2 u^2 + \mu) \psi_1 = 0 \quad (3.32)$$

$$\frac{d^2 \psi_2}{dv^2} + (k^2 v^2 - \mu) \psi_2 = 0 \quad (3.33)$$

respectively.

Thus the Lamé equations in the contraction limit go over into equations (3.32), (3.33) for parabolic cylinder functions [36]. The same is of course true for solutions. The expansion (3.13) is not suitable for the contraction limit. In view of (2.52) we need expansions in terms of the variables  $(1 + \operatorname{sn} \alpha)$  and  $(1 - \sqrt{2} \operatorname{dn} \beta)$ . This is not hard to do, following for instance methods used in [37] to relate the wavefunctions of a two-dimensional hydrogen atom, calculated in different coordinate systems. The formulae are cumbersome, so we shall not present them here.

## 4. Conclusions

In this paper we have presented a new aspect of the theory of Lie group and Lie algebra contractions: the relation between separable coordinate systems in curved and flat spaces, related by the contraction of their isometry groups. So far we have considered the simplest meaningful example, namely the original Wigner–Inonu contraction from a  $O(3)$  to  $E(2)$ , as applied to the sphere  $S_2$  and Euclidean plane  $E_2$ .

We have followed through the contraction  $R \rightarrow \infty$  (where  $R$  is the radius of the sphere  $S_2$ ) at all levels: the Lie algebras as realized by vector fields, the Laplace–Beltrami operators in the two spaces, the second-order operators in the enveloping algebras, characterizing

separable systems, the separable coordinate systems themselves, the separated (ordinary) differential equations and the separated eigenfunctions of the invariant operators.

In particular, we have shown how different limiting procedures lead from two separable systems on  $S_2$ , to four systems on the plane  $E_2$ .

The contraction parameter in this paper was a geometrical one, namely the radius  $R$  of the  $S_2$  sphere. The results of this paper are also applicable in a completely different context, namely that of high energy physics. Indeed, an elementary particle can be described by a wavefunction, transforming according to an irreducible unitary representation of the Poincare group [38, 39]. Its spin is associated with the Pauli–Lubanski vector

$$W_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} M^{\nu\lambda} P^\rho .$$

If the particle is massive, the linear momentum is timelike  $p^2 = m^2 > 0$ . In the rest frame we have  $p_0 = m$ ,  $\mathbf{p} = 0$  and the space components of  $W_\mu$  generate an  $\mathfrak{o}(3)$  algebra. If the particle is massless, then its momentum is lightlike,  $p^2 = 0$  and no rest frame exists. We can choose a frame with  $p = (\omega, 0, 0, \omega)$  and the three linearly independent components of  $W_\mu$  generate an  $\mathfrak{e}(2)$  algebra. Let us parametrize the linear momentum of a massive particle in spherical coordinates on the mass shell

$$\begin{aligned} p_0 &= m \cosh a & p_1 &= m \sinh a \sin \theta \cos \phi \\ p_2 &= m \sinh a \sin \theta \sin \phi & p_3 &= m \sinh a \cos \theta . \end{aligned}$$

At very high energies  $p_0 \rightarrow \infty$  we have

$$\cosh a \sim \sinh a \sim \frac{1}{2} e^a$$

and the momentum  $p$  approaches a lightlike one.

All results of this paper are applicable in this case with a reinterpretation of the contraction parameter. This role is now played by the particle energy  $p_0$ , or more precisely by the ‘energy’ coordinate  $a$ . The physical relevance of such a contraction is obvious: in the ultrarelativistic limit [40] massive particles start to behave like massless ones, for which spin considerations, for example, reduce to helicity ones.

We mention that the realization (2.26)–(2.28) of  $\mathfrak{o}(3)$  in terms of vector fields in two variables corresponds to the  $O(3)$  subgroup of the group  $SL(3, \mathbb{R})$  figuring as the group of projective transformations of  $\mathbb{R}^2$ . Indeed, more generally,  $\mathfrak{sl}(n + 1, \mathbb{R})$  can be realized as

$$\begin{aligned} P_\mu &= \frac{\partial}{\partial x_\mu} & L_{\mu\nu} &= x_\mu \frac{\partial}{\partial x_\nu} & C_\mu &= x_\mu D \\ D &\equiv \sum_{\alpha=1}^n x_\alpha \frac{\partial}{\partial x_\alpha} & 1 &\leq \mu, \quad \nu \leq n. \end{aligned} \tag{4.1}$$

This shows that the methods of this paper can easily be adapted to treat other contraction problems. For  $n = 2$  we can construct the  $\mathfrak{o}(2, 1) \subset \mathfrak{sl}(3, \mathbb{R})$  subalgebra and consider contractions of separable coordinates on one- or two-sheeted hyperboloids and their contractions to coordinates on Euclidean or pseudo-Euclidean planes. For  $n \geq 3$  we can again extract  $\mathfrak{o}(n)$ , or  $\mathfrak{o}(p, q)$  subalgebras of  $\mathfrak{sl}(n, \mathbb{R})$  and consider similar contraction problems. This, however, lies beyond the scope of the present paper.

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## References

- [1] Vilenkin N Ya 1968 *Special Functions and the Theory of Group Representation* (Providence, RI: American Mathematical Society)
- [2] Talman J D 1968 *Special Functions: A Group Theoretic Approach* (New York: Benjamin)
- [3] Miller W Jr 1977 *Lie Theory and Special Functions* (New York: Academic)
- [4] Exton H 1983 *q-Hypergeometric Functions and Applications* (Chichester: Ellis Horwood)
- [5] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)
- [6] Vaksman L L and Soibelman Ya S 1988 *Funct. Anal. Pril.* **22** 1
- [7] Noumi M and Minachi K 1990 *Commun. Math. Phys.* **128** 521
- [8] Floreanini R and Vinet L 1992 *J. Math. Phys.* **33** 1358
- [9] Winternitz P and Fris I 1965 *Sov. J. Nucl. Phys.* **1** 636
- [10] Winternitz P, Lukac I and Smorodinskii Ya A 1968 *Sov. J. Nucl. Phys.* **7** 139
- [11] Miller W Jr 1977 *Symmetry and Separation of Variables* (Reading, MA: Addison-Wesley)
- [12] Kalnins E G 1986 *Separation of Variables for Riemannian Spaces of Constant Curvature* (London: Longman)
- [13] Miller W Jr, Patera J and Winternitz P 1991 *J. Math. Phys.* **22** 251
- [14] Pogosyan G S, Smorodinsky Ya A and Ter-Antonyan V M 1981 *J. Phys. A: Math. Gen.* **14** 769
- [15] Mardoyan L G, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 1985 *J. Phys. A: Math. Gen.* **18** 455
- [16] Rideau G and Winternitz P 1993 *J. Math. Phys.* **34** 6030
- [17] Inonu E and Wigner E P 1953 *Proc. Natl Acad. Sci. USA* **39** 510
- [18] Saletan E J 1961 *J. Math. Phys.* **2** 1
- [19] Gilmore R 1974 *Lie Groups, Lie Algebras and Some of their Applications* (New York: Wiley)
- [20] Moody R V and Patera J 1991 *J. Phys. A: Math. Gen.* **24** 2227
- [21] De Montigny M and Patera J 1991 *J. Phys. A: Math. Gen.* **24** 525
- [22] Couture M, Patera J, Sharp R T and Winternitz P 1991 *J. Math. Phys.* **32** 2310
- [23] Celeghini E, Giachetti E and Tarlini M 1992 *Contractions of Quantum Groups (Lecture Notes in Mathematics 1510)* (Berlin: Springer) p 221
- [24] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1993 *J. Phys. A: Math. Gen.* **26** 5801
- [25] Hussin V, Lauzon A and Rideau G 1994 *Lett. Math. Phys.* **2** 159
- [26] Patera J and Winternitz P 1973 *J. Math. Phys.* **14** 1130
- [27] Lukac I and Smorodinsky Ya A 1970 *Sov. Phys.-JETP* **30** 728
- [28] Lukac I 1973 *Theor. Math. Phys.* **14** 271
- [29] Lukac I and Smorodinsky Ya A 1973 *JINR Preprint P2-7465*, Dubna
- [30] Hobson E W 1965 *The Theory of Spherical and Ellipsoidal Harmonics* (New York: Chelsea)
- [31] Byrd R F and Friedman M D 1971 *Handbook of Elliptic Integrals for Engineers and Scientists* (Berlin: Springer)
- [32] Kadyshevsky V G 1962 *Dokl. Akad. Nauk. SSSR* **147** 588
- [33] Varshalovich D A, Moskalev A N and Khersonskii V K 1975 *Quantum Theory of Angular Momentum* (Leningrad: Nauka)
- [34] Arscott F M 1964 *Periodic Differential Equations* (New York: Macmillan)
- [35] Bateman H and Erdélyi 1955 *Higher Transcendental Functions* vol 3 (New York: McGraw-Hill)
- [36] Bateman H and Erdélyi 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill)
- [37] Mardoyan L G, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 1984 *Theor. Math. Phys.* **61** 1021
- [38] Wigner E P 1939 *Ann. Math.* **40** 149
- [39] Bargmann V and Wigner E P 1948 *Proc. Natl Acad. Sci. USA* **34** 211
- [40] De Alfaro V, Fubini S, Furlan G and Rosetti C 1973 *Currents in Hadron Physics* (Amsterdam: North-Holland/New York: American Elsevier)